Grasp Verification

Definitions

- **Grasp**: A grasp can be fully specified with description of the object and the contact points. More specifically, a grasp is completely defined when following two entities are known:
  - Object Geometry
  - Set of contact points and corresponding contact normals
- **Grasp Stability**: Ability of a grasp to resist any arbitrary external force. In other words, a grasp is stable if it can produce contact forces that can resist any arbitrary disturbance

Problem

The problem at hand is to find out if a particular grasp is stable or not.

Approach

In reality, a contact point will not be a single point. Also, it will always have some amount of friction along with the normal force. However, we ignore both these facts to simplify the analysis and assume that all the contacts are frictionless point contacts.

A simple case of this problem with 1-D object was analyzed in a previous lecture. As a next step, a 2-D object is considered here.

![Figure 1: A 2D object, showing co-ordinate axes, contact points and contact normals](image)

In the case of 2-D objects, unit normal force at every contact point is defined as a 2x1 vector consisting of x and y components of the normal force. For the four contact points shown in
Figure 1, the unit normal forces are as follows:

\[ \hat{n}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \hat{n}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad \hat{n}_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \hat{n}_4 = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \]

For 2-D objects, analysis of only the forces is not enough since a moment along the z-axis can also be applied externally. The grasp needs to resist any external moment along with resistance to external forces. To find resultant moment from every contact force, we need the positions of point of contact.

\[ \vec{r}_1 = \begin{bmatrix} -2 \\ -1 \end{bmatrix}, \quad \vec{r}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \vec{r}_3 = \begin{bmatrix} 0 \\ -2 \end{bmatrix}, \quad \vec{r}_4 = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \]

With these coordinates at hand, moment of a contact force can be calculated as:

\[ M_i = \vec{r}_i \times \hat{n}_i \]

In order to analyze the stability of grip, we will use a compact representation of forces and moments lined up in a single vector. This vector is known as the 'Wrench matrix'. For 2-D objects, the wrench matrix for a contact is defined as a 3x1 vector with x and y components of the unit normal contact force and moment of the unit normal contact force \([n_x \ n_y \ M]^T\). The concept of wrench matrix can be easily extended to 3-D objects with a 6x1 vector of \([n_x \ n_y \ n_z \ M_x \ M_y \ M_z]^T\)

For the contact forces in Figure 1, wrench matrices will be as follows:

\[ W_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad W_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad W_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad W_4 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \]

Thus, a grip will be a stable grip if these contact forces can resist any arbitrary external wrench. The constraint here is that the contact forces can only push into the object and cannot pull it. So, scaling of individual wrenches is possible only with non-negative scaling factors.

In order to test if an arbitrary external force can be countered by given set of contacts, we first map the individual wrenches on wrench space. Wrench space is defined as mutually orthogonal space with unit component vectors along every dimension of a wrench. For a 2-D objects, wrench space is 3-D space with \(F_x, F_y\) and \(M\) as three mutually orthogonal dimensions of the space. The wrench space for 3-D objects will be 6-D space with \(F_x, F_y, F_z, M_x, M_y\) and \(M_z\) as six mutually orthogonal dimensions.
Figure 2 shows the mapping of four contact wrenches from the contacts as shown in Figure 1. As said before, each of these wrenches can be scaled by an arbitrary positive number. The resultant wrench of all four contacts will be sum of individual scaled wrenches. Figure 3 shows the part of total wrench space that can be covered by arbitrarily scaling the four vectors from Figure 2. It is apparent from this figure that this set of contacts cannot cover the entire wrench space.

Since the entire wrench space is not covered by the set of contacts, the grasp shown in Figure 1 is NOT stable. Specifically, this grip cannot resist an external wrench such that $-W_{ext}$ lies
beyond the bound space as highlighted in Figure 3 (\(-W_{ext}\) is the resultant wrench to be applied by at the points to resists external wrench). The simplest example where this grip fails is when external wrench consists only of a unit counterclockwise moment.

\[
W_{ext} = \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}
\]

To counter this wrench, the grip needs to produce a resultant wrench equal and opposite to \(W_{ext}\).

\[
W_{target} = -W_{ext} = \begin{bmatrix}
0 \\
0 \\
-1
\end{bmatrix}
\]

Since none of the contact wrenches can produce negative moment, it is impossible to achieve this target wrench with the grasp as shown in Figure 1.

Intuitively, by observing the wrench matrices of these contacts, we can produce forces and moments along \(+F_x, -F_x, +F_y, -F_y\) and \(+M\) directions, but there is no way to produce moment along \(-M\) direction. To stabilize this grip, we need to change the contacts in such a way that they can produce forces and moments along all six directions. To achieve the only missing direction, viz. \(-M\), we can shift contact points 3 and 4 so that they can produce a clockwise moment (clockwise moment is negative). Figure 4 shows this configuration.

![Figure 4: Adjusted contact points to provide force closure](image-url)
For this grip:

\[
\begin{align*}
\hat{n}_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \hat{n}_2 &= \begin{bmatrix} -1 \\ 0 \end{bmatrix} & \hat{n}_3 &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \hat{n}_4 &= \begin{bmatrix} 0 \\ -1 \end{bmatrix} \\
\vec{r}_1 &= \begin{bmatrix} -2 \\ -1 \end{bmatrix} & \vec{r}_2 &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} & \vec{r}_3 &= \begin{bmatrix} -1 \\ -2 \end{bmatrix} & \vec{r}_4 &= \begin{bmatrix} 1 \\ 2 \end{bmatrix}
\end{align*}
\]

And

\[
\begin{align*}
W_1 &= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} & W_2 &= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} & W_3 &= \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} & W_4 &= \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}
\end{align*}
\]

Thus, this grip can produce forces and moments along all six directions in wrench space. This can be verified visually by coverage of these wrench matrices in Figure 5.

Figure 5: Updated wrench space coverage map, showing force closure

Once we figure out all the wrench matrices at all the contact points, we then define a **Grasp map**, which is a matrix in which all the wrenches are written as columns.

\[
G = \begin{bmatrix} W_1 & W_2 & \ldots & W_n \end{bmatrix}, \text{ where } W_i \text{ is the } i^{th} \text{ Wrench Matrix}
\]
\( \lambda = [\lambda_1 \quad \lambda_2 \quad \ldots \quad \lambda_n]^T \), where \( \lambda_i \) is the scale for the wrench matrix \( W_i \). Notice that the \( \lambda \) matrix is a column vector.

The above representation helps us in a couple of ways:

- Firstly, we will be able to see the span of the forces that we can apply by using the properties of this matrix, from linear algebra tools such as rank, etc.

- Secondly, it provides us with an intuitive way to represent the forces, since the \( \lambda \) matrix will denote the kind of forces that we can apply (positive being only push, etc.), and the resultant will be the linear combination of the wrenches, each of them multiplied by the scale \( \lambda_i \). As such, we can also see at a glance which contact force contributes how much.

The resultant wrench applied by the grasp will be:

\[
W = G \ast \lambda = W_1 \ast \lambda_1 + W_2 \ast \lambda_2 + \ldots + W_n \ast \lambda_n
\]

Since this should negate the external wrench \( W_{\text{ext}} \), we want to have

\[
G \ast \lambda = -W_{\text{ext}}
\]

Now, we need to find tools that will help us know if our Grasp map and the \( \lambda \) values are sufficient to overcome any external force.

**Numerical methods for grasp stability analysis**

1. Mathematically speaking, in order to overcome an arbitrary external force, we want our \( G \ast \lambda \) values to span all the possible dimensions of the external forces. By this, we mean that if we are in a 2-D scenario, then the number of dimensions we need to be concerned about is 3 (2 for forces and one for moment, represented in the dimension of the wrench matrix). Since in our case we are concerned about push forces (\( \lambda_i >= 0 \)), we can look into the rank of the \( G \) matrix to get the necessary information.

2. One other way to ensure the stability of the grasp is to take the wrench matrices as vertices, take the convex hull of these vertices, and check if the origin is in the interior of the convex hull.

   This works because our wrench vectors represent the unit forces along the dimensions, and the convex hull of these as vertices represents the \( L_1 \) grasp wrench space, as described by Ferrari and Canny (C. Ferrari and J. Canny. Planning optimal grasps. In Proc. of the 1992 IEEE Intl. Conf. on Robotics and Automation, pages 2290–2295, 1992.). The idea is that if the origin is inside, we will have the opportunity to generate necessary internal forces that can act against the external force, but still can add up to zero having no effect on the body.
There are many algorithms to find out the convex hull of a given set of points. One of them in the 2-D case is the graham scan, as described in http://en.wikipedia.org/wiki/Graham_scan

We can also look at the MATLAB inbuilt function such as `convhull()` and `inpolygon()` to find out the convex hull, and to check if the point is inside the polygon respectively.

Examples:

We will see how the analysis of internal forces helps us in the following examples. Note that it is assumed in these examples that only push forces are applicable:

![Figure 6: 1-D example for force closure](image)

In figure 6, we can see that the only way to oppose a force applied at $x_1$ would be to apply an equal and opposite force, which is possible by the application at $x_2$, and since we cannot create any moment by applying forces at these points, we need not consider moments in analysis for this example. As such, this configuration is stable.

![Figure 7: 2-D configuration with contact normals applied to the center of the object](image)

We can see in figure 7 that any forces applied at $x_1$ and $y_1$ can be negated by applying a negating forces on $x_2$ and $y_2$ without generating a moment. This means that it is possible for us to have internal forces that can resist any kind of force (not moment) in this configuration.
The configuration in figure 8 shows the necessity for the mathematical tools. We can see that if a force is applied at \( x_1 \), the only way to bring the force in \( x \) direction would be to have an equal and opposite force at \( x_2 \). But this means that we have a moment applied on the body, which cannot be negated by applying forces at \( y_1 \) and \( y_2 \), since they apply forces about the center of the body, which cannot create a moment.

So, the only way we can have the internal forces add up to zero would be to apply 0 forces in the \( x \) direction, and equal and opposite forces in \( y \), which gives the \( \lambda \) matrix as \([0 \ 0 \ a \ a]^T; \ a \geq 0\). Here we cannot create internal forces at all contact points and this case represents the scenario where the origin is on the border of the convex hull (as shown in Figure. 3), and not in the interior, thereby not giving us the full rank.

The above example shows the case where the origin will be in the interior of the convex hull. The contact points are configured in such a way that we can also negate the moments by applying the forces in the right way.

One sample \( \lambda \) matrix is \([1 \ 1 \ 1 \ 1]^T\) (assuming that the forces are at symmetric distances, on all four sides), which gives us the intuition that we get the full rank.

We can of course prove this to ourselves by finding the convex hull, which is similar to that shown in Figure 5 above. We can clearly notice that the origin is in the interior of the hull.
Coulomb Friction

2-D Coulomb Friction

To begin to understand how friction affects our model, we must first re-develop our model for friction. The most common model used is Coulomb Friction.

![Free body diagram of a mass on an incline, showing coulomb friction forces](image)

Figure 10: Free body diagram of a mass on an incline, showing coulomb friction forces

The weight of the body can be divided into two components as:

\[
F_n = Mg \cdot \cos(\theta) \\
F_t = Mg \cdot \sin(\theta)
\]

If we increase theta to the angle just before it slips:

\[
\tan(\theta_{\text{max}}) = \frac{F_t}{F_n}
\]

The angle \(\theta_{\text{max}}\) is called the angle of repose, and the tangent of this angle is called the coefficient of static friction, \(\mu\).

More generally in 2D, you can make a plot of all the possible tangential and normal directional forces a contact can exert.
Looking at figure 11, all possible contact forces are represented as being within the triangle shown above where it should be clear that the borders are described as $\mathbf{n}_1 = \begin{bmatrix} \sin(\theta_{\text{max}}) \\ \cos(\theta_{\text{max}}) \end{bmatrix}$ and $\mathbf{n}_2 = \begin{bmatrix} -\sin(\theta_{\text{max}}) \\ \cos(\theta_{\text{max}}) \end{bmatrix}$.

Notice that if $\theta_{\text{max}} = 0$, a frictionless surface, $\mathbf{n}_1$ and $\mathbf{n}_2$ collapse on a single vector, the normal vector, implying that the contact force can only exert forces along the normal.

From this picture, it should be clear that the contact force could be represented a few ways. The first of which is a positive linear combination of $\mathbf{n}_1$ and $\mathbf{n}_2$ that looks like:

$$f = \lambda_1 \mathbf{n}_1 + \lambda_2 \mathbf{n}_2,$$

where $\lambda_1, \lambda_2 \geq 0$

Next, in the tangential and normal representation:

$$f = [F_t F_n]^T$$

subject to $F_n \geq 0$ and $\tan\left(\frac{|F_t|}{F_n}\right) \leq \mu$
3D Coulomb Friction:

The 2-D Coulomb Friction model can easily be extended into 3D by revolving the plot of tangential vs normal friction and about the normal axis.

The 3D model works much the same as the 2D model, where the valid contact forces simply lay within the cone rather than within the triangle.