1. **Effect of Friction at contact points**

The forces considered at the contact points were limited to the normal forces up to this point. The introduction of friction at contact points improves the grasp of the object. Friction provides tangential force to resist external forces in the direction perpendicular to normal force exerted by the contact points on the object.

In 2D space, the resultant force due to normal force and friction force can be represented by a friction triangle bounded by unit vectors $\hat{n}_1$ and $\hat{n}_2$ as shown in figure 7, having force magnitudes $\lambda_1$ and $\lambda_2$ respectively. Hence, the effective force is represented by

$$f = \lambda_1 n_1 + \lambda_2 n_2 \text{ where } \lambda_1, \lambda_2 \geq 0$$

![Fig. 1 2D Friction cone](image)

Similarly, in 3D space, the friction triangle turns into a friction cone as shown in figures 2 and 3, with frictional forces in the plane represented by $\hat{t}_1$ and $\hat{t}_2$ and the normal force perpendicular to them represented by $\hat{n}$. The net force is 3D space is represented by

$$f = \begin{bmatrix} f_{t1} \\ f_{t2} \\ f_n \end{bmatrix}$$
To evaluate the forces experienced by the grasped object in 3D space, the cone can be approximated to a polyhedron. For simplifying the solution, we will consider a polyhedron with 4 sides as shown in figure 4. Let $\alpha_i$ represent the angular positions on the cone where we are placing the vertices for approximation. If we divide the cone into four equal sides, we obtain

$$\alpha_i = \frac{2\pi_i}{N} \quad \text{where } i = 1, 2, 3, \ldots N$$

$$N = 4 \quad i = 1, i = 2, i = 3, i = 4$$

$$\alpha_i: \frac{2\pi (1)}{4}, \frac{2\pi (2)}{4}, \frac{2\pi (3)}{4}, \frac{2\pi (4)}{4}$$

Consider vectors $T_i$, which are unit vectors along the direction along the vertices of the polyhedron from the center of the top planar surface of the cone.
Fig. 4 Top view of 3D Friction cone estimated to a polyhedron of 4 sides

\[ T_i = \begin{bmatrix} \cos \alpha_i \\ \sin \alpha_i \end{bmatrix}, |T_i| = 1 \]

Hence, the effective force vectors at the point of contact of the polyhedron will be given by

\[ n_i = \begin{bmatrix} \sin \theta_{\text{max}} T_i \\ \cos \theta_{\text{max}} \end{bmatrix} \]

This leads us to the representation of the coefficient of friction

\[ \frac{\sin \theta_{\text{max}} T_i}{\cos \theta_{\text{max}}} = \tan \theta_{\text{max}} = \mu \]

Let us generalize this concept for any M number of vertices selected on the cone’s surface for evaluating the forces

**2D Coulomb friction:** \( n_1, n_2 \)

**3D Coulomb friction:** \( n_1, n_2, n_3 \)

The unit wrenches for each face of the polyhedron will be given by

\[ w_i = \begin{bmatrix} n_i \\ T_i \times n_i \end{bmatrix} \]

And the corresponding Grasp Map is given by

\[ G = [w_1, w_2 \ldots w_{NM}] \]
2. **Conditions for forced closure**

a) G is full rank
b) Origin is in the interior of the wrench convex hull

These conditions are necessary but not sufficient for forced closure.
For example, consider the figure below. The contact points of the grasp are as shown in the figure. If a unit internal force is applied at each of these contact points, the total internal force is reduced to zero. But, a rotational force applied to this body will not be resisted due to the cancellation of all the internal forces at the center of the body. In this case, the origin lies inside the body, but the grasp is not firm.

![Diagram showing contact points and forces](image)

**Fig. 5** Box enclosed using the given 4 contact points with zero internal force but no forced closure

\[
\begin{align*}
\mathbf{w}_1 &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} & \mathbf{w}_2 &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} & \mathbf{w}_3 &= \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} & \mathbf{w}_4 &= \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}
\end{align*}
\]

Grasp Quality Measures

There are several methods available in literature to determine the grasp quality. Following are the measures based on the grasp matrix G.

1. **Least Singular value**

   Geometrically if a sphere is multiplied by a matrix it results in an ellipsoid (figure 6). So if a sphere in torque domain is multiplied by the matrix G it will be mapped to an ellipsoid in the force domain or wrench space. Following are the metrics for the quality of grasp:
The parameters of the ellipsoid geometry determine the quality of grasp. If we do the Singular Value Decomposition of the matrix $G$ we get:

$$SVD(G) = U\Sigma V^T$$

From the decomposed matrices the matrix $\Sigma$ contains the singular values along the diagonal i.e. $\sigma_1$ and $\sigma_2$. When a grasp is in a singular configuration (i.e. when at least one degree of freedom is lost due to hand configuration), at least one of the singular values goes to zero. The singular values give the measure of the geometry or shape of the ellipsoid. The smaller of the two values quantifies how far the grasp is from miscarriage. Larger values of the $\sigma_2$ corresponds to better grasps.

2. Volume of Ellipsoid

Another metric for quality of grasp is the volume of the ellipsoid, the larger the ellipsoid. For instance of two wrench spaces have ellipsoids with same values of sigma but one with larger volume the one with larger volume will be termed as a more secure grasp as it encompasses more space. The volume is given by:

$$\sqrt{\text{Det}(G G^T)}$$
3. **Isotropy Index:**

It is the ratio of the minimum singular value to the maximum singular value of the grasp matrix $G$:

\[
\frac{\sigma_{\text{min}}}{\sigma_{\text{max}}} \approx 1
\]

It is a measure of the uniformity of wrench forces that are applied to an object. If the value of the isotropy index is one geometrically it corresponds to a sphere implying that all the forces apply uniformly from all sides. The smaller it is the more ellipsoidal it will be.

In case of any arbitrary shaped object the corresponding transformation in the wrench space will not be ellipsoidal rather it will be some random shape (figure 7). To analyze the grasp in such case we can fit an inscribed ellipsoid inside the shape and gage the grasp quality by the above metrics.

![Fig. 7 Transformation of an arbitrary shaped object to wrench space](image)

**Grasp Tables**

Trying different grasp solutions by simulating physics and planning is computationally expensive. To avoid online computations this processing is done offline to generate a table which maps grasp solutions for different types of hands to different types of objects. Samples from this solution set are applied online and checked for successful grasps. This sampling is not just random. It is done such that a diversified set of solutions is tried to increase the probability of a successful grasp.

Let us analyze a 2-dimensional problem in which HERB has to grasp an object with his Barret hand (figure 8). The hand parameters that can be controlled are the angle of approach, the aperture of hand, the x and y position of the hand.

\[
\begin{bmatrix}
x \\
y \\
\theta \\
a
\end{bmatrix}
\]

![Fig. 8 Barret hand grasping an object (2-d case)](image)

![Fig. 9 Parameters of the Barret hand](image)
Following are the steps to reach a grasp solution (figure 10):

- Angle of approach is chosen
- The hand aperture is set
- The object is approached to make a contact
- The hand is closed to achieve a grasp

The grasp matrix is then calculated and the resultant ellipsoid in the wrench space is analyzed to test the quality of grasp. Another method is to bring the shape of the target object into consideration. Depending upon the shape the grasps can be chosen (figure 11) but the drawback is that the diversity is compromised and in some cases the planner can end up finding no solution. For example in the problem shown below based upon the geometry of the cup the planner may choose to grasp the cup from under the table which is not possible (figure 12).

Another technique that is also used to generate grasps is to decompose the object model into simplified known geometric shapes (figure 13) and the approximation works fine in many cases but still it cannot be standardized or generalized as there is no set rules to do that and the decomposition solutions may vary.