A Unifying Formalism for Shortest Path Problems with Expensive Edge Evaluations via Lazy Best-First Search over Paths with Edge Selectors

Christopher M. Dellin and Siddhartha S. Srinivasa
The Robotics Institute, Carnegie Mellon University
{cdellin, siddh}@cs.cmu.edu

Abstract
While the shortest path problem has myriad applications, the computational efficiency of suitable algorithms depends intimately on the underlying problem domain. In this paper, we focus on domains where evaluating the edge weight function dominates algorithm running time. Inspired by approaches in robotic motion planning, we define and investigate the Lazy Shortest Path class of algorithms which is differentiated by the choice of an edge selector function. We show that several algorithms in the literature are equivalent to this lazy algorithm for appropriate choice of this selector. Further, we propose various novel selectors inspired by sampling and statistical mechanics, and find that these selectors outperform existing algorithms on a set of example problems.

1 Introduction
Graphs provide a powerful abstraction capable of representing problems in a wide variety of domains from computer networking to puzzle solving to robotic motion planning. In particular, many important problems can be captured as shortest path problems (Figure 1), wherein a path $p^*$ of minimal length is desired between two query vertices through a graph $G$ with respect to an edge weight function $w$.

Despite the expansive applicability of this single abstraction, there exist a wide variety of algorithms in the literature for solving the shortest path problem efficiently. This is because the measure of computational efficiency, and therefore the correct choice of algorithm, is inextricably tied to the underlying problem domain.

The computational costs incurred by an algorithm can be broadly categorized into three sources corresponding to the blocks in Figure 1. One such source consists of queries on the structure of the graph $G$ itself. The most commonly discussed such operation, expanding a vertex (determining its successors), is especially fundamental when the graph is represented implicitly, e.g. for domains with large graphs such as the 15-puzzle or Rubik’s cube. It is with respect to vertex expansions that $A^*$ (Hart, Nilsson, and Raphael 1968) is optimally efficient.

A second source of computational cost consists of maintaining ordered data structures inside the algorithm itself, which is especially important for problems with large branching factors. For such domains, approaches such as partial expansion (Yoshizumi, Miura, and Ishida 2000) or iterative deepening (Korf 1985) significantly reduce the number of vertices generated and stored by either selectively filtering surplus vertices from the frontier, or by not storing the frontier at all.

The third source of computational cost arises not from reasoning over the structure of $G$, but instead from evaluating the edge weight function $w$ (i.e. we treat discovering an out-edge and determining its weight separately). Consider for example the problem of articulated robotic motion planning using roadmap methods (Kavraki et al. 1996). While these graphs are often quite small (fewer than $10^5$ vertices), determining the weight of each edge requires performing many collision and distance computations for the complex geometry of the robot and environment, resulting in planning times of multiple seconds to find a path.

In this paper, we consider problem domains in which evaluating the edge weight function $w$ dominates algorithm running time and investigate the following research question:

How can we minimize the number of edges we need to evaluate to answer shortest-path queries?

We make three primary contributions. First, inspired by lazy collision checking techniques from robotic motion planning (Bohlin and Kavraki 2000), we formulate a class of shortest-path algorithms that is well-suited to problem domains with expensive edge evaluations. Second, we show that several existing algorithms in the literature can be expressed as special cases of this algorithm. Third, we show that the extensibility afforded by the algorithm allows for novel edge evaluation strategies, which can outperform existing algorithms over a set of example problems.
Algorithm 1 Lazy Shortest Path (LazySP)

1: function LAZYSHORTESTPATH(G, u, w, west)
2: \( E_{eval} \leftarrow \emptyset \)
3: \( w_{lazy}(e) \leftarrow w_{est}(e) \quad \forall e \in E \)
4: loop
5: \( p_{candidate} \leftarrow \text{SHORTESTPATH}(G, u, w_{lazy}) \)
6: if \( p_{candidate} \subseteq E_{eval} \) then
7:     return \( p_{candidate} \)
8: end if
9: \( E_{selected} \leftarrow \text{SELECTOR}(G, p_{candidate}) \)
10: for \( e \in E_{selected} \setminus E_{eval} \) do
11:     \( w_{lazy}(e) \leftarrow w(e) \) \triangleright Evaluate (expensive)
12: end for
13: \( E_{eval} \leftarrow E_{eval} \cup e \)
14: end loop

Algorithm 2 Various Simple LazySP Edge Selectors

1: function SELECTEXPAND(G, \( p_{candidate} \))
2: \( e_{first} \leftarrow \text{first unevaluated } e \in p_{candidate} \)
3: \( v_{frontier} \leftarrow G.\text{source}(e_{first}) \)
4: \( E_{selected} \leftarrow G.\text{out-edges}(v_{frontier}) \)
5: return \( E_{selected} \)

6: function SELECTFORWARD(G, \( p_{candidate} \))
7: return \{first unevaluated \( e \in p_{candidate} \}\)

8: function SELECTREVERSE(G, \( p_{candidate} \))
9: return \{last unevaluated \( e \in p_{candidate} \}\)

10: function SELECTALTERNATE(G, \( p_{candidate} \))
11: if LazySP iteration number is odd then
12:     return \{first unevaluated \( e \in p_{candidate} \}\)
13: else
14:     return \{last unevaluated \( e \in p_{candidate} \}\)
15: end if
16: return \{unevaluated \( e \in p_{candidate} \}\)

Theorem 1 (Completeness of LazySP) If the graph \( G \) is finite, \( \text{SHORTESTPATH} \) is complete, and the set \( E_{selected} \) returned by \( \text{SELECTOR} \) returns at least one unevaluated edge on \( p_{candidate} \), then \( \text{LAZYSHORTESTPATH} \) is complete.

Theorem 2 (Optimality of LazySP) If \( w_{est} \) is chosen such that \( w_{est}(e) \leq \epsilon w(e) \) for some parameter \( \epsilon \geq 1 \) and \( \text{LAZYSHORTESTPATH} \) terminates with some path \( \epsilon_{pret} \), then \( \text{len}(\epsilon_{pret}, w) \leq \epsilon \ell^* \) with \( \ell^* \) the length of an optimal path.

The optimality of LazySP depends on the admissibility of \( w_{est} \) in the same way that the optimality of \( A^* \) depends on the admissibility of its goal heuristic \( h \). Theorem 2 establishes the general bounded suboptimality of LazySP w.r.t. the inflation parameter \( \epsilon \). While our theoretical results (e.g. equivalences) hold for any choice of \( \epsilon \), for clarity our examples and experimental results focus on cases with \( \epsilon = 1 \).

The Edge Selector: Key to Efficiency

The LazySP algorithm exhibits a rough similarity to optimal replanning algorithms such as \( D^* \) (Stentz 1994) which plan a sequence of shortest paths for a mobile robot as new edge weights are discovered during its traversal. \( D^* \) treats edge changes passively as an aspect of the problem setting (e.g. a sensor with limited range).

The key difference is that our problem setting treats edge evaluations as an active choice that can be exploited. While any choice of edge selector that meets the conditions above will lead to an algorithm that is complete and optimal, its efficiency is dictated by the choice of this selector. This motivates the theoretical and empirical investigation of different edge selectors in this paper.

Simple selectors. We codify five common strategies in Algorithm 2. The Expand selector captures the edge weights that are evaluated during a conventional vertex expansion. The selector identifies the first unevaluated edge \( e_{first} \) on the
candidate path, and considers the source vertex of this edge a *frontier* vertex. It then selects all out-edges of this frontier vertex for evaluation. The Forward and Reverse selectors select the first and last unevaluated edge on the candidate path, respectively (note that Forward returns a subset of Expand).

The Alternate selector simply alternates between Forward and Reverse on each iteration. This can be motivated by both bidirectional search algorithms as well as motion planning algorithms such as RRT-Connect (Kuffner and LaValle 2000) which tend to perform well w.r.t. state evaluations.

The Bisection selector chooses among those unevaluated edges the one furthest from an evaluated edge on the candidate path. This selector is roughly analogous to the collision checking strategy employed by the Lazy PRM algorithm from (Bohlin and Kavraki 2000) as applied to our shortest path problem on abstract graphs.

In the following section, we demonstrate that instances of LazySP using simple selectors yield equivalent results to existing vertex algorithms. The subsequent section then discusses two more sophisticated selectors motivated by weight function sampling and statistical mechanics.

## 3 Edge Equivalence to A* Variants

In the previous section, we introduced LazySP as the path-selection analogue to BFS vertex-selection algorithms. In this section, we make this analogy more precise. In particular, we show that LazySP-Expand is edge-equivalent to a variant of A* (and Weighted A*), and that LazySP-Forward is edge-equivalent to a variant of Lazy Weighted A* (see Table 1). It is important to be specific about the conditions under which these equivalences arise, which we detail here.

### Table 1: LazySP equivalence results. The A*, LWA*, and BHFFA algorithms use reopening and the dynamic \( h_{\text{lazy}} \) heuristic (4).

<table>
<thead>
<tr>
<th>LazySP Selector</th>
<th>Existing Algorithm</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expand</td>
<td>(Weighted) A*</td>
<td>Edge-equivalent (Theorems 3, 4)</td>
</tr>
<tr>
<td>Forward</td>
<td>Lazy Weighted A*</td>
<td>Edge-equivalent (Theorems 5, 6)</td>
</tr>
<tr>
<td>Alternate</td>
<td>Bidirectional Heuristic Front-to-Front Algorithm</td>
<td>Conjectured</td>
</tr>
</tbody>
</table>

**Edge equivalence.** We say that two algorithms are *edge-equivalent* if they evaluate the same edges in the same order. We consider an algorithm to have evaluated an edge the first time the edge’s true weight is requested.

**Arbitrary tiebreaking.** For some graphs, an algorithm may have multiple allowable choices at each iteration (e.g. LazySP with multiple candidate shortest paths, or A* with multiple vertices in OPEN with lowest \( f \)-value). We will say that algorithm A is equivalent to algorithm B if for any choice available to A, there exists an allowable choice available to B such that the same edge(s) are evaluated by each.

**A* with reopening.** We show equivalence to variants of A* and Lazy Weighted A* that do not use a CLOSED list to prevent vertices from being visited more than once.

**A* with a dynamic heuristic.** In order to apply A* and Lazy Weighted A* to our problem, we need a goal heuristic.
over vertices. The most simple may be
\[
h_{\text{est}}(v) = \min_{p : v \rightarrow v_g} \text{len}(p, w_{\text{est}}).
\] (3)

Note that the value of this heuristic could be computed as a pre-processing step using Dijkstra’s algorithm (Dijkstra 1959) before iterations begin. However, in order for the equivalences to hold, we require the use of the lazy heuristic
\[
h_{\text{lazy}}(v) = \min_{p : v \rightarrow v_g} \text{len}(p, w_{\text{lazy}}).
\] (4)

This heuristic is dynamic in that it depends on \(w_{\text{lazy}}\) which changes as edges are evaluated. Therefore, heuristic values must be recomputed for all affected vertices on OPEN after each iteration.

**Equivalence to A***

We show that the LazySP-Expand algorithm is edge-equivalent to a variant of the A* shortest-path algorithm. We make use of two invariants that are maintained during the progression of A*. Proofs are provided in the appendix.

**Invariant 1** If \(v\) is discovered by A* and \(v'\) is undiscovered, with \(v'\) a successor of \(v\), then \(v\) is on OPEN.

**Invariant 2** If \(v\) and \(v'\) are discovered by A*, with \(v'\) a successor of \(v\), and \(g[v] + w(v, v') < g[v']\), then \(v\) is on OPEN.

When we say a vertex is discovered, we mean that it is either on OPEN or CLOSED. Note that Invariant 2 holds because we allow vertices to be reopened; without reopening (and with an inconsistent heuristic), later finding a cheaper path to \(v\) (and not reopening \(v\)) would invalidate the invariant.

We will use the goal heuristic \(h_{\text{lazy}}\) from (4). Note that if an admissible edge weight estimator \(\hat{w}\) exists (that is, \(\hat{w} \leq w\)), then our A* can approximate the Weighted A* algorithm (Pohl 1970) with parameter \(\epsilon\) by using \(w_{\text{est}} = \epsilon \hat{w}\), and the suboptimality bound from Theorem 2 holds.

**Equivalence.** In order to show edge-equivalence, we consider the case where both algorithms are beginning a new iteration having so far evaluated the same set of edges.

LazySP-Expand has some set \(P_{\text{candidate}}\) of allowable candidate paths minimizing \(\text{len}(p, w_{\text{lazy}})\); the Expand selector will then identify a vertex on the chosen path for expansion.

A* will iteratively select a set of vertices from OPEN to expand. Because it is possible that a vertex is expanded multiple times (and only the first expansion results in edge evaluations), we group iterations of A* into sequences, where each sequence \(s\) consists of (a) zero or more vertices from OPEN that have already been expanded, followed by (b) one vertex from OPEN that is to be expanded for the first time.

We show that both the set of allowable candidate paths \(P_{\text{candidate}}\) available to LazySP-Expand and the set of allowable candidate vertex sequences \(S_{\text{candidate}}\) available to A* map surjectively to the same set of unexpanded frontier vertices \(V_{\text{frontier}}\) as illustrated in Figure 3. This is described by way of Theorems 3 and 4 below (proofs in appendix).

**Theorem 3** If LazySP-Expand and A* have evaluated the same set of edges, then for any candidate path \(P_{\text{candidate}}\), chosen by LazySP yielding frontier vertex \(v_{\text{frontier}}\), there exists an allowable A* sequence \(S_{\text{candidate}}\) which also yields \(v_{\text{frontier}}\).

**Theorem 4** If LazySP-Expand and A* have evaluated the same set of edges, then for any candidate sequence \(S_{\text{candidate}}\), chosen by A* yielding frontier vertex \(v_{\text{frontier}}\), there exists an allowable LazySP path \(P_{\text{candidate}}\) which also yields \(v_{\text{frontier}}\).

**Equivalence to Lazy Weighted A***

In a conventional vertex expansion algorithm, determining a successor’s cost is a function of both the cost of the edge and the value of the heuristic. If either of these components is expensive to evaluate, an algorithm can defer its computation by maintaining the successor on the frontier with an approximate cost until it is expanded. The Fast Downward algorithm (Helmer 2006) is motivated by expensive heuristic evaluations in planning, whereas the Lazy Weighted A* (LWA*) algorithm (Cohen, Phillips, and Likhachev 2014) is motivated by expensive edge evaluations in robotics.

We show that the LazySP-Forward algorithm is edge-equivalent to a variant of the Lazy Weighted A* shortest-path algorithm. For a given candidate path, the Forward selector returns the first unevaluated edge.

**Variant of Lazy Weighted A***. We reproduce a variant of LWA* without a CLOSED list in Algorithm 3. For the purposes of our analysis, the reproduction differs from the original presentation, and we detail those differences here. With the exception of the lack of CLOSED, the differences do not affect the behavior of the algorithm.

**Algorithm 3 Lazy Weighted A* (without CLOSED list)**

1: \textbf{function} \textsc{LazyWeightedA}(*(G, w, \hat{w}, h)*)
2: \(g[v_{\text{start}}] \leftarrow 0\)
3: \(Q_e \leftarrow \{v_{\text{start}}\} \quad \triangleright \text{Key: } g[v] + h(v)\)
4: \(Q_e \leftarrow \emptyset \quad \triangleright \text{Key: } g[v] + \hat{w}(v, v') + h(v')\)
5: \textbf{while} \(\min(Q_e, \text{TopKey}, Q_e, \text{TopKey}) < g[v_{\text{goal}}]\) \textbf{do}
6: \textbf{if} \(Q_e, \text{TopKey} \leq Q_e, \text{TopKey}\) \textbf{then}
7: \(v \leftarrow Q_e, \text{Pop}()\)
8: \textbf{for} \(v' \in G\).GetSuccessors(\(v\)) \textbf{do}
9: \(Q_e, \text{Insert}((v, v'))\)
10: \textbf{else}
11: \((v, v') \leftarrow Q_e, \text{Pop}()\)
12: \textbf{if} \(g[v'] \leq g[v] + \hat{w}(v, v')\) \textbf{then}
13: \textbf{continue}
14: \(g_{\text{new}} \leftarrow g[v] + \hat{w}(v, v')\) \quad \triangleright \text{evaluate}
15: \textbf{if} \(g_{\text{new}} < g[v']\) \textbf{then}
16: \(g[v'] = g_{\text{new}}\)
17: \(Q_e, \text{Insert}(v')\)
The most obvious difference is that we present the original OPEN list as separate vertex \((Q_v)\) and edge \((Q_e)\) priority queues, with sorting keys shown on lines 3 and 4. A vertex \(v\) in the original OPEN with \(trueCost(v) = true\) corresponds to a vertex \(v\) in \(Q_v\); whereas a vertex \(v'\) in the original OPEN with \(trueCost(v') = false\) (and parent \(v\)) corresponds to an edge \((v, v')\) in \(Q_e\). Use of the edge queue obviates the need for (a) duplicate vertices on OPEN with different parents, and (b) the \(conf(v)\) test for identifying such duplicates. This presentation also highlights the similarity between LWA* and the inner loop of the Batch Informed Trees (BIT*) algorithm (Gammell, Srinivasa, and Barfoot 2015).

The second difference is that the test for an edge’s usefulness (line 12 of the original algorithm) has been moved from before inserting into OPEN to after being popped from OPEN, but before being evaluated (line 12 of Algorithm 3). This change is partially in compensation for removing the CLOSED list. Note that moving the test later (but before the edge is evaluated) does not affect the number or order of edge evaluations.

We make use of an invariant that is maintained during the progression of Lazy Weighted A* (proof in appendix).

**Invariant 3** For all vertex pairs \(v, v'\), with \(v'\) a successor of \(v\), if \(g[v] + max (\hat{w}(v, v'), \hat{w}(v, v')) < g[v']\), then either \(v\) is on \(Q_v\) or edge \((v, v')\) is on \(Q_e\).

We will use \(h(v) = h_{lazy}(v)\) from (4) and \(\hat{w} = \hat{w}_{lazy}\). Note that the use of these dynamic heuristics requires that the \(Q_v\) and \(Q_e\) be resorted after every edge is evaluated.

**Equivalence.** The equivalence follows similarly to that for A* above. Given the same set of edges evaluated, the set of allowable next evaluations is identical for each algorithm.

**Theorem 5** If LazySP-Forward and LWA* have evaluated the same set of edges, then for any allowable candidate path \(p_{candidate}\) chosen by LazySP yielding first unevaluated edge \(e_{ab}\), there exists an allowable LWA* sequence \(s_{candidate}\) which also yields \(e_{ab}\).

**Theorem 6** If LazySP-Forward and LWA* have evaluated the same set of edges, then for any allowable sequence of vertices and edges \(s_{candidate}\) considered by LWA* yielding evaluated edge \(e_{ab}\), there exists an allowable LazySP candidate path \(p_{candidate}\) which also yields \(e_{ab}\).

**Relation to Bidirectional Heuristic Search**

LazySP-Alternate chooses unevaluated edges from either the beginning or the end of the candidate path at each iteration. We conjecture that an alternating version of the Expand selector is edge-equivalent to the Bidirectional Heuristic Front-to-Front Algorithm (Sint and de Champeaux 1977) for appropriate lazy vertex pair heuristic, and that LazySP-Alternate is edge-equivalent to a bidirectional LWA*.

4 Novel Edge Selectors

Because we are conducting a search over paths, we are free to implement selectors which are not constrained to evaluate edges in any particular order (i.e. to maintain evaluated edges which also yield the same set of edges, then for any allowable candidate path \(p_{candidate}\) chosen by LazySP yielding first unevaluated edge \(e_{ab}\), there exists an allowable LWA* sequence \(s_{candidate}\) which also yields \(e_{ab}\).

**Algorithm 4 Maximum Edge Probability Selector**

(for WeightSamp and Partition path distributions)

1: \(\text{function} \ SELECT_{MAX_{EDGE\_PROB}}(G, p_{candidate}, D_p)\)
2: \(p(e) \leftarrow \Pr (e \in P) \) for \(P \sim D_p\)
3: \(e_{\text{max}} \leftarrow \text{unevaluated } e \in p_{candidate} \text{ maximizing } p(e)\)
4: \(\text{return } \{e_{\text{max}}\}\)

![Algorithm 4](image)

**Figure 4:** Illustration of maximum edge probability selectors. A distribution over paths (usually conditioned on the known edge evaluations) induces on each edge \(e\) a Bernoulli distribution with parameter \(p(e)\) giving the probability that it belongs to the path. The selector chooses the edge with the largest such probability.

**Algorithm 4 Maximum Edge Probability Selector**

(for WeightSamp and Partition path distributions)

1: \(\text{function} \ SELECT_{MAX_{EDGE\_PROB}}(G, p_{candidate}, D_p)\)
2: \(p(e) \leftarrow \Pr (e \in P) \) for \(P \sim D_p\)
3: \(e_{\text{max}} \leftarrow \text{unevaluated } e \in p_{candidate} \text{ maximizing } p(e)\)
4: \(\text{return } \{e_{\text{max}}\}\)

**Figure 4:** Illustration of maximum edge probability selectors. A distribution over paths (usually conditioned on the known edge evaluations) induces on each edge \(e\) a Bernoulli distribution with parameter \(p(e)\) giving the probability that it belongs to the path. The selector chooses the edge with the largest such probability.

**Algorithm 4 Maximum Edge Probability Selector**

(for WeightSamp and Partition path distributions)

1: \(\text{function} \ SELECT_{MAX_{EDGE\_PROB}}(G, p_{candidate}, D_p)\)
2: \(p(e) \leftarrow \Pr (e \in P) \) for \(P \sim D_p\)
3: \(e_{\text{max}} \leftarrow \text{unevaluated } e \in p_{candidate} \text{ maximizing } p(e)\)
4: \(\text{return } \{e_{\text{max}}\}\)

**Algorithm 4 Maximum Edge Probability Selector**

(for WeightSamp and Partition path distributions)

1: \(\text{function} \ SELECT_{MAX_{EDGE\_PROB}}(G, p_{candidate}, D_p)\)
2: \(p(e) \leftarrow \Pr (e \in P) \) for \(P \sim D_p\)
3: \(e_{\text{max}} \leftarrow \text{unevaluated } e \in p_{candidate} \text{ maximizing } p(e)\)
4: \(\text{return } \{e_{\text{max}}\}\)

**Algorithm 4 Maximum Edge Probability Selector**

(for WeightSamp and Partition path distributions)

1: \(\text{function} \ SELECT_{MAX_{EDGE\_PROB}}(G, p_{candidate}, D_p)\)
2: \(p(e) \leftarrow \Pr (e \in P) \) for \(P \sim D_p\)
3: \(e_{\text{max}} \leftarrow \text{unevaluated } e \in p_{candidate} \text{ maximizing } p(e)\)
4: \(\text{return } \{e_{\text{max}}\}\)

**Algorithm 4 Maximum Edge Probability Selector**

(for WeightSamp and Partition path distributions)

1: \(\text{function} \ SELECT_{MAX_{EDGE\_PROB}}(G, p_{candidate}, D_p)\)
2: \(p(e) \leftarrow \Pr (e \in P) \) for \(P \sim D_p\)
3: \(e_{\text{max}} \leftarrow \text{unevaluated } e \in p_{candidate} \text{ maximizing } p(e)\)
4: \(\text{return } \{e_{\text{max}}\}\)

**Algorithm 4 Maximum Edge Probability Selector**

(for WeightSamp and Partition path distributions)

1: \(\text{function} \ SELECT_{MAX_{EDGE\_PROB}}(G, p_{candidate}, D_p)\)
2: \(p(e) \leftarrow \Pr (e \in P) \) for \(P \sim D_p\)
3: \(e_{\text{max}} \leftarrow \text{unevaluated } e \in p_{candidate} \text{ maximizing } p(e)\)
4: \(\text{return } \{e_{\text{max}}\}\)

**Algorithm 4 Maximum Edge Probability Selector**

(for WeightSamp and Partition path distributions)

1: \(\text{function} \ SELECT_{MAX_{EDGE\_PROB}}(G, p_{candidate}, D_p)\)
2: \(p(e) \leftarrow \Pr (e \in P) \) for \(P \sim D_p\)
3: \(e_{\text{max}} \leftarrow \text{unevaluated } e \in p_{candidate} \text{ maximizing } p(e)\)
4: \(\text{return } \{e_{\text{max}}\}\)

**Algorithm 4 Maximum Edge Probability Selector**

(for WeightSamp and Partition path distributions)

1: \(\text{function} \ SELECT_{MAX_{EDGE\_PROB}}(G, p_{candidate}, D_p)\)
2: \(p(e) \leftarrow \Pr (e \in P) \) for \(P \sim D_p\)
3: \(e_{\text{max}} \leftarrow \text{unevaluated } e \in p_{candidate} \text{ maximizing } p(e)\)
4: \(\text{return } \{e_{\text{max}}\}\)
Scores after five evaluations

Algorithm 4 can then be written:

\[ D_p : f_p(p) \propto \exp(-\beta \text{len}(p, w_{\text{lazy}})). \]  

(6)

In other words, we consider all potential paths \( P \) between the start and goal vertices, with shorter paths assigned more probability than longer ones (with positive parameter \( \beta \)).

We call this the Partition selector because this distribution is closely related to calculating partition functions from statistical mechanics. The corresponding partition function is:

\[ Z(P) = \sum_{p \in P} \exp(-\beta \text{len}(p, w_{\text{lazy}})). \]  

(7)

Note that the edge indicator probability required in Algorithm 4 can then be written:

\[ p(e) = 1 - \frac{Z(P \setminus e)}{Z(P)}. \]  

(8)

Here, \( P \setminus e \) denotes paths in \( P \) that do not contain edge \( e \).

It may appear advantageous to restrict the set \( P \) to only simple paths, since all possible problems with \( w \geq 0 \) admit an optimal path that is simple. Unfortunately, an algorithm for computing (8) efficiently is not currently known in this case. However, in the case that \( P \) consists of all paths, there exists an efficient incremental calculation of (7) via a recursive formulation which we detail here.

For brevity, we use the notation \( Z_{xy} = Z(P_{xy}) \), with \( P_{xy} \) the set of all paths from vertex \( x \) to \( y \). Suppose the values \( Z_{xy} \) are known between all pairs of vertices \( x, y \) for a graph \( G \). (For a graph with no edges, \( Z_{xy} \) is 1 if \( x = y \) and 0 otherwise.) Consider a modified graph \( G' \) with one additional edge \( e_{ab} \) with weight \( w_{ab} \). Since all additional paths use the new edge \( e_{ab} \), a non-zero number of times, the value of \( Z'_{xy} \) can be shown to be:

\[ Z'_{xy} = Z_{xy} + \frac{Z_{xa} Z_{hy}}{\exp(\beta w_{ab}) - Z_{ba}} \text{ if } \exp(\beta w_{ab}) > Z_{ba}. \]  

(9)

This form is derived from simplifying the induced geometric series; note that if \( \exp(\beta w_{ab}) \leq Z_{ba} \), the value \( Z'_{xy} \) is infinite. It is also possible to derive the inverse relationship; that is, given values \( Z' \), calculate the values \( Z \) if an edge with a given weight were removed.

This incremental formulation of (7) allows for the corresponding score \( p(e) \) for edges to be updated efficiently during each iteration of LazySP as the \( w_{\text{lazy}} \) value for edges chosen for evaluation are updated. In fact, if the values \( Z \) are stored in a square matrix, the update for all pairs after an edge weight change consists of a single vector outer product.
5 Experiments

We compared the seven edge selectors on three classes of shortest path problems, and collected the average number of edges evaluated by each, shown in Figure 8. In each case, the estimate was chosen so that $w_{est} \leq w$, so that all runs produced optimal paths. The experimental results serve primarily to illustrate that the A* and LWA* algorithms (i.e. Expand and Forward) are not optimally efficient w.r.t. edge evaluations, but also expose differences in behavior and point to potential areas for future research.

All experiments were conducted using an open-source implementation of the LazySP algorithm and selectors.\footnote{Link to implementation hidden for double-blind review.} Motion planning results were implemented using OMPL (Sucan, Moll, and Kavraki 2012).

Random partially-connected graphs. We tested on a set of 1000 randomly-generated undirected graphs with $|V| = 100$, with each pair of vertices sharing an edge with probability 0.05. Edges have an independent 0.5 probability of having infinite weight, else the weight is uniformly distributed on $[1, 2]$; the estimated weight was unity for all edges. For the WeightSamp selector, we drew 1000 $w$ samples at each iteration from the above edge weight distribution. For the Partition selector, we used $\beta = 2$.

Roadmap graphs on the unit square. We considered roadmap graphs formed via the first 100 points of the $(2, 3)$-Halton sequence on the unit square with a connection radius of 0.15, with 30 pairs of start and goal vertices chosen randomly. The edge weight function was derived from 30 sampled obstacle fields consisting of 10 randomly placed axis-aligned boxes with dimensions uniform on $[0.1, 0.3]$, with each edge having infinite weight on collision, and weight equal to its Euclidean length estimate otherwise. One of the resulting 900 example problems is shown in Figure 2. For the WeightSamp selector, we drew 1000 $w$ samples with a naïve edge weight distribution in which each edge had an independent 0.1 probability of being in collision. For the Partition selector, we used $\beta = 25$.

Roadmap graphs for robot arm motion planning. We considered roadmap graphs in the configuration space corresponding to right arm of the HERB home robot across three motion planning problems inspired by a table clearing scenario (see Figure 7). The joint extents of each of the robot’s seven degrees of freedom range from 3.12 to 6.09 rad. The problems consisted of first moving from the robot’s home configuration to one of 7 feasible grasp configurations for a mug (pictured), second transferring the mug to one of 72 feasible configurations with the mug above the blue bin, and third returning to the home configuration. Each problem was solved independently. This common scenario spans various numbers of starts/goals and allows a comparison w.r.t. difficulty at different problem stages as discussed later.

For each problem, 50 random graphs were constructed by applying a random offset to the 7D Halton sequence with $N = 1000$, with additional vertices for each problem start and goal configuration. We used an edge connection radius of 3 rad, resulting $|E|$ ranging from 23404 to 28109. Each edge took infinite weight on collision, and weight equal to its Euclidean length otherwise. For the WeightSamp selector, we drew 1000 $w$ samples with a naïve edge weight distribution in which each edge had an independent 0.1 probability of collision. For the Partition selector, we used $\beta = 3$.

6 Discussion

The first observation that is evident from the experimental results is that lazy evaluation – whether using Forward (LWA*) or one of the other selectors – grossly outperforms Expand (A*). As one might expect, the relative penalty that Expand incurs by evaluating all edges from each expanded vertex is correlated with the graph’s branching factor.

Since the Forward and Reverse selectors are simply mirrors of each other, they exhibit identical performance averaged across the ParConn and UnitSquare problem classes, which are symmetric. However, this need not the case for a particular problem. For example, the start of ArmPlan1 and the goal of ArmPlan3 both consist of the arm’s single home configuration in a relatively confined space. As shown in the table in Figure 8a, it appears that the better selector on these problems attempts to solve the more constrained side of the problem first. While it may be difficult to determine a priori which part of the problem will be the most constrained, the simple Alternate selector’s respectable performance suggests that it may be a reasonable compromise.

The per-path plots at the bottom of Figure 2 allow us to
members of many potential paths. Because it tends to focus evaluations in a similar way, the Alternate selector may serve as a simple proxy for the more complex selectors.

We note that an optimal selector could be theoretically achieved by posing the edge selection problem as a POMDP, given an accurate probabilistic model of the true costs. While likely intractable for larger graphs in complex domains, exploring this solution may yield useful approximations or insights.

Optimizations. While this paper has focused on edge evaluations as the dominant source of computational cost, other considerations may also be important, especially on larger graphs. There are a number of optimizations that allow for efficient implementation of LazySP.

The first relates to the repeated invocations of the inner shortest path algorithm (line 5 of Algorithm 1). Because only a small number of known edges change lazy edge weights between invocations, an incremental search algorithm such as SSSP (Ramalingam and Reps 1996) or LPA* (Koenig, Likhachev, and Furcy 2004) can be used to greatly improve the speed of the inner searches. Since the edge selector determines where on the graph edges are evaluated, the choices of the selector and the inner search algorithm are related. For example, using the Forward selector with an incremental inner search rooted at the goal results in behavior similar to D* (Stentz 1994) (albeit without the need to handle a moving start location) since a large portion of the inner tree can be reused upon every forward evaluation.

An optimization commonly applied to vertex searches called immediate expansion is also applicable to LazySP. If an edge is evaluated with weight \( w \leq w_{\text{est}} \), the inner search need not be run again before the next edge is evaluated.

A third optimization is applicable to domains with infinite edge costs (e.g. to represent infeasible transitions). If the length of the path returned by the inner shortest path algorithm is infinite, LazySP can return this path immediately even if some of its edges remain unexplored without affecting its (sub)optimality. This reduces the number of edge evaluations needed in the case that no feasible path exists.

Other methods for expensive edge evaluations. An alternative to lazy evaluations is based on the observation that when solved by vertex expansion algorithms, such problems are characterized by slow vertex expansions. To mitigate this, approaches such as Parallel A* (Irani and Shih 1986) and Parallel A* for Slow Expansions (Phillips, Koenig, and Likhachev 2014) aim to parallelize such expansions. We believe that a similar approach can be applied to LazySP.

Another approach to finding short paths quickly is to relax the optimization objective (2) itself. While LazySP already supports a bounded-suboptimal objective via an inflated edge weight estimate (Theorem 2), it may also be possible to adapt the algorithm to address bounded-cost problems (Stern, Puzis, and Felner 2011).
References


A Appendix: Proofs

LazySP

Proof of Theorem 2 Let \( p^* \) be an optimal path w.r.t. \( w, \) with \( \ell^* = \text{len}(p^*, w) \). Since \( w_{\text{eval}}(e) \leq \epsilon w(e) \) and \( \epsilon \geq 1 \), it follows that regardless of which edges are stored in \( W_{\text{eval}} \), \( w_{\text{lazy}}(e) \leq \epsilon w(e) \), and therefore \( \text{len}(p^*, w_{\text{lazy}}) \leq \epsilon \ell^* \). Now, since the inner \textsc{ShortestPath} algorithm terminated with \( p_{\text{ret}} \), we know that \( \text{len}(p_{\text{ret}}, w_{\text{lazy}}) \leq \text{len}(p^*, w_{\text{lazy}}) \). Further, since the algorithm terminated with \( p_{\text{ret}} \), each edge on \( p_{\text{ret}} \) has been evaluated; therefore, \( \text{len}(p_{\text{ret}}, w) = \text{len}(p_{\text{ret}}, w_{\text{lazy}}) \). Therefore, \( \text{len}(p_{\text{ret}}, w) \leq \epsilon \ell^* \). □

Proof of Theorem 1 In this case, the algorithm will evaluate at least unevaluated edge at each iteration. Since there are a finite number of edges, eventually the algorithm will terminate. □

A* Equivalence

Proof of Invariant 1 If \( v \) is discovered, then it must either be on \( \text{OPEN} \) or \( \text{CLOSED} \). \( v \) can be on \( \text{CLOSED} \) only after it has been expanded, in which case \( s \) would be discovered (which is it not). Therefore, \( v \) must be on \( \text{OPEN} \). □

Proof of Invariant 2 Clearly the invariant holds at the beginning of the algorithm, with only \( v_{\text{start}} \) on \( \text{OPEN} \). If the invariant were to no longer hold after some iteration, then there must exist some pair of discovered vertices \( v \) and \( v' \) with \( v \) on \( \text{CLOSED} \) and \( g[v] + w(v, v') < g[v'] \). Since \( v \) is on \( \text{CLOSED} \), it must have been expanded at some previous iteration, immediately after which the inequality could not have held because \( g[v'] \) is updated upon expansion of \( v \). Therefore, the inequality must have newly held after some intervening iteration, with \( v \) remaining on \( \text{CLOSED} \). Since the values \( g \) are monotonically non-increasing and \( w \) is fixed, this implies that \( g[v] \) must have been updated (lower). However, if this had happened, then \( v \) would have been removed from \( \text{CLOSED} \) and placed on \( \text{OPEN} \). This contradiction implies that the invariant holds at every iteration. □

Proof of Theorem 3 Consider path \( p^{\ast}_{\text{lazy}} \) with length \( \ell^{\ast}_{\text{lazy}} \) yielding frontier vertex \( v_{\text{frontier}} \) via \textsc{SelectExpand}. Construct a vertex sequence \( s \) as follows. Initialize \( s \) with the vertices on \( p^{\ast}_{\text{lazy}} \) from \( v_{\text{start}} \) to \( v_{\text{frontier}} \), inclusive. Let \( N \) be the number of consecutive vertices at the start of \( s \) for which \( f(v) = \ell^{\ast}_{\text{lazy}} \) (Note that the first vertex on \( p^{\ast}_{\text{lazy}} \), \( v_{\text{start}} \), must have \( f(v_{\text{start}}) = \ell^{\ast}_{\text{lazy}} \), so \( N \geq 1 \)). Remove from the start of \( s \) the first \( N - 1 \) vertices. Note that at most the first vertex on \( s \) has \( f(v) = \ell^{\ast}_{\text{lazy}} \), and the last vertex on \( s \) must be \( v_{\text{frontier}} \).

Now we show that each vertex in this sequence \( s \), considered by A* in turn, exists on \( \text{OPEN} \) with minimal \( f \)-value. Iteratively consider the following procedure for sequence \( s \). Throughout, we know that there must not be any vertex with \( f(v) < \ell^{\ast}_{\text{lazy}} \), that would imply that a different path through \( v \) shorter than \( \ell^{\ast}_{\text{lazy}} \) exists, in which case \( p^{\ast}_{\text{lazy}} \) could not have been chosen.

If the sequence has length \( > 1 \), then consider the first two vertices on \( s \), \( v_a \) and \( v_b \). By construction, \( f(v_a) = \ell^{\ast}_{\text{lazy}} \) and \( f(v_b) = \ell^{\ast}_{\text{lazy}} \). In fact, from above we know that \( f(v_b) > \ell^{\ast}_{\text{lazy}} \). Therefore, we have that \( f(v_a) < f(v_b) \), therefore and \( g[v_a] + w(v_a, v_b) < g[v_b] \). By Invariant 2, \( v_a \) must be on \( \text{OPEN} \), and with \( f(v_a) = \ell^{\ast}_{\text{lazy}} \), it can therefore be considered by A*. After it is expanded, \( f(v_b) = \ell^{\ast}_{\text{lazy}} \), and we can repeat the above procedure with the sequence formed by removing the \( v \) from \( s \).

If instead the sequence has length 1, then it must be exactly \( (v_{\text{frontier}}, v_{\text{frontier}}) = (\ell^{\ast}_{\text{lazy}}, \ell^{\ast}_{\text{lazy}}) \). Since the edge after \( f(v_{\text{frontier}}) \) is not evaluated, then by Invariant 1, \( v_{\text{frontier}} \) must be on \( \text{OPEN} \), and will therefore be expanded next. □

Proof of Theorem 4 Given that all vertices in \( \text{SCandidate} \) besides the last are re-expansions, they can be expanded with no edge evaluations. Once the last vertex, \( v_{\text{frontier}} \), is to be expanded by A*, suppose it has \( f \)-value \( \ell \).

First, we will show that there exists a path with length \( \ell \) w.r.t. \( w_{\text{lazy}} \) wherein all edges before \( v_{\text{frontier}} \) have been evaluated, and the first edge after \( v_{\text{frontier}} \) has not. Let \( p_a \) be a shortest path from \( v_{\text{start}} \) to \( v_{\text{frontier}} \) consisting of only evaluated edges. The length of this \( p_a \) must be equal to \( g[v_{\text{frontier}}] \); if it were not, there would be some previous vertex on \( p_a \) with lower \( f \)-value than \( v_{\text{frontier}} \), which would necessarily have been expanded first. Let \( p_b \) be the a shortest path from \( v_{\text{frontier}} \) to \( v_{\text{goal}} \). The length of \( p_b \) must be \( \ell_{\text{lazy}}(v_{\text{frontier}}) \) by definition. Therefore, the path \( (p_a, p_b) \) must have length \( \ell \), and since \( v_{\text{frontier}} \) is a new expansion, the first edge on \( p_b \) must be unevaluated.

Second, we will show that there does not exist any path shorter than \( \ell \) w.r.t. \( w_{\text{lazy}} \). Suppose \( p' \) were such a path, with length \( \ell' < \ell \). Clearly, \( v_{\text{start}} \) would have \( f \)-value \( \ell' \) (although it may not be on \( \text{OPEN} \)). Consider each pair of vertices \( (v_a, v_b) \) along \( p' \) in turn. In each case, if \( v_b \) were either undiscovered, or if \( g[v_a] + w(v_a, v_b) < g[v_b] \), then \( v_a \) would be on \( \text{OPEN} \) (via Invariants 1 and 2, respectively) with \( f(v_a) = \ell' \), and would therefore have been expanded before \( v_{\text{frontier}} \). Otherwise, we know that \( f(v_b) = \ell' \), and we can continue to the next pair on \( p' \). □

LWA* Equivalence

Proof of Invariant 3 Clearly the invariant holds at the beginning of the algorithm with only \( g[v_{\text{start}}] = 0 \), since the inequality holds only for the out-edges of \( v_{\text{start}} \) with \( v_{\text{start}} \) on \( Q_e \). Consider each subsequent iteration. If a vertex \( v \) is popped from \( Q_e \), then this may invalidate the invariant for all successors of \( v \); however, since all out-edges are immediately added to \( Q_e \), the invariant must hold. Consider instead if an edge \( (v, v') \) which satisfies the inequality is popped from \( Q_e \). Due to the inequality, we know that \( g[v'] \) will be recalculated as \( g[v'] = g[v] + w(v, v') \), so that the inequality is no longer satisfied for edge \( (v, v') \). However, reducing the value \( g[v'] \) may introduce satisfied inequalities across subsequent out-edges of \( v' \), but since \( v' \) is added to \( Q_e \), the invariant continues to hold. □

Proof of Theorem 5 In the first component of the equivalence, we will show that for any path \( p \) minimizing \( w_{\text{lazy}} \) allowable to LazySP-Forward, with \( (v_a, v_b) \) the first unevaluated edge on \( p \), there exists a sequence of vertices and edges on \( Q_e \) and \( Q_c \) allowable to LWA* such that edge \( (v_a, v_b) \) is
the first to be newly evaluated. Let the length of \( p \) w.r.t. \( w_{\text{lazy}} \) be \( \ell' \).

We will first show that no vertex on \( Q_v \) or edge on \( Q_e \) can have \( f(\cdot) < \ell \). Suppose such a vertex \( v \), or edge \( e \) with source vertex \( v \), exists. Then \( g[v] + h(v) < \ell \), and there must be some path \( p' \) consisting of an evaluated segment from \( v_{\text{start}} \) to \( v \) of length \( g[v] \), followed a segment from \( v \) to \( v_{\text{goal}} \) of length \( h(v) \). But then this path should have been chosen by LazySP.

Next, we will show a procedure for generating an allowable sequence for LWA*. We will iteratively consider a sequence of path segments, starting with the segment from \( v_{\text{start}} \) to \( v_a \), and becoming progressively shorter at each iteration by removing the first vertex and edge on the path.

We will show that the first vertex on each segment \( v_f \) has \( g[v_f] = \ell - h(v_f) \). By definition, this is true of the first such segment, since \( g[v_{\text{start}}] = 0 \). For each but the last such segment, consider the first edge, \( (v_f, v_e) \). If \( v_e \) has the correct \( g[\cdot] \), we can continue to the next segment immediately. Otherwise, either \( v_f \) is on \( Q_v \), or \( (v_f, v_e) \) is on \( Q_e \) by Invariant 3. If the former is true, then \( v_f \) can be popped from \( Q_v \) with \( f = \ell \), thereby adding \( (v_f, v_e) \) to \( Q_e \). Then, \( (v_f, v_e) \) can be popped from \( Q_e \) with \( f = \ell \), resulting in \( g[v_e] = \ell - h(v_e) \). We can then move on to the next segment.

At the end of this process, we have the trivial segment \( (v_a) \), with \( g[v_a] = \ell - h(v_a) \). If \( v_a \) is on \( Q_v \), then pop it (with \( f(v_a) = \ell \)), placing \( e_{ab} \) on \( Q_e \); otherwise, since \( e_{ab} \) is unevaluated, it must already be on \( Q_e \). Since \( f(e_{ab}) = \ell \), we can pop and evaluate it.

**Proof of Theorem 6** Given that all vertices in \( s_{\text{candidate}} \) entail no edge evaluations, and all edges therein are re-expansions, they can be considered with no edge evaluations. Once the last edge \( e_{xy} \) is to be expanded by LWA*, suppose it has \( f\)-value \( \ell \).

First, we will show that there exists a path with length \( \ell \) w.r.t. \( w_{\text{lazy}} \) which traverses unevaluated edge \( e_{xy} \) wherein all edges before \( v_x \) have been evaluated. Let \( p_x \) be a shortest path segment from \( v_{\text{start}} \) to \( v_x \) consisting of only evaluated edges. The length of \( p_x \) must be equal to \( g[v_x] \); if it were not, there would be some previous vertex on \( p_x \) with lower \( f\)-value than \( v_x \), which would necessarily have been expanded first. Let \( p_y \) be the a shortest path from \( v_y \) to \( v_{\text{goal}} \). The length of \( p_y \) must be \( h_{\text{lazy}}(v_y) \) by definition. Therefore, the path \( (p_x, e_{xy}, p_y) \) must have length \( \ell \).

Second, we will show that there does not exist any path shorter than \( \ell \) w.r.t. \( w_{\text{lazy}} \). Suppose \( p' \) were such a path, with length \( \ell' < \ell \), and with first unexpanded edge \( e_{xy}' \). Clearly, \( v_{\text{start}} \) has \( g[v_{\text{start}}] = \ell' - h(v_{\text{start}}) \). Consider each evaluated edge \( e_{xy}' \) along \( p' \) in turn. In each case, if \( g[v_y] \neq \ell' - h(v_y) \), then \( e_{xy}' \) would be on \( Q_e \) or \( Q_v \) with \( f(\cdot) = \ell' \), and would therefore be expanded before \( e_{xy} \). Therefore, \( e_{xy}' \) would then be popped from \( Q_v \) with \( f(e_{xy}') = \ell' \), and it would have been evaluated before \( e_{xy} \) with \( f(e_{xy}) = \ell \).